On a Vertex-Capturing Game

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Abstract

In this paper, we study the recently introduced scoring game played on graphs called the *Edge-Balanced Index Game*. This game is played on a graph by two players, Alice and Bob, who take turns colouring an uncoloured edge of the graph. Alice plays first and colours edges red, while Bob colours edges blue. The game ends once all the edges have been coloured. A player *captures* a vertex if more than half of its incident edges are coloured by that player, and the player that captures the most vertices wins.

Using classical arguments from the field, we first prove general properties of this game. Namely, we prove that there is no graph in which Bob can win (if Alice plays optimally), while Alice can never capture more than 2 more vertices than Bob (if Bob plays optimally). Through dedicated arguments, we then investigate more specific properties of the game, and focus on its outcome when played in particular graph classes. Specifically, we determine the outcome of the game in paths, cycles, complete bipartite graphs, and Cartesian grids, and give partial results for trees and complete graphs.

Keywords: scoring game; combinatorial game; 2-player game; graph.

1. Introduction

In this work, we study a 2-player scoring game played on graphs called the *Edge-Balanced Index Game* [5]. Initially, we were unaware that this game was already introduced in [5], and we were actually inspired to study this game by the board game Kahuna designed by Günter Cornett and first published by Kosmos in 1998. Let us start by giving an overview of the main features of Kahuna, which is a turn-based 2-player board game. On the board, there are 12 islands, and some of them are connected by bridges. The game includes cards, which the players can draw at the beginning of the game or at the end of each turn. Each card depicts one of the 12 islands. During a turn, a player can take a certain number of actions, the main of which is to play successive cards, each showing an island, and, for each such island, claim an unclaimed bridge going from that island to a neighbouring one (*i.e.*, connected by the bridge). Whenever claiming a bridge, a player can capture one of the two islands it joins, this being possible only if they have claimed more than half of its connecting bridges. Whenever a player captures an island, all the bridges that were claimed by the opponent are withdrawn, which, in turn, can have the cascading effect of making the opponent lose its control over neighbouring islands, and so on. Due to these mechanisms, note that, during the course of a game, islands can be repeatedly

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captured by either of the two players or by none of them. The game ends once there are no more cards to be drawn, and the winner is the player that, eventually, has captured the most islands.

The main intent of this work is to study the primary mechanisms behind Kahuna through a 2-player scoring game played on graphs called the Edge-Balanced Index Game. Note that this makes sense, as there is definitely a natural graph structure underlying the game, as the islands can be modelled as the vertices of a graph, while the bridges can be modelled as the edges of that graph. Due to some of the board game features, it would not be reasonable to model them all in a single game on graphs, as it would make the analysis much more complicated and uncertain. More precisely, the Edge-Balanced Index Game models a more static version of the game, in which the card-drawing mechanism is dropped (and thus, the random aspects of the game), and the fact that claimed bridges can be lost by the players during the course of a game is also dropped. Instead, the game is an impartial one where the players may claim any unclaimed bridges one at a time, and it is a scoring game since the players' scores are computed (and thus, the outcome) once all the bridges have been claimed.

The precise rules of the Edge-Balanced Index Game are as follows. The game is played on an undirected connected[2] graph G. The edges of G are initially uncoloured. Successive rounds take place, during each of which, a first player called Alice, colours an uncoloured edge of G red, before a second player called Bob, colours an uncoloured edge (if any remain) blue. The game ends once all the edges of G have been coloured. To decide the outcome of the game, the *scores* of the two players are then calculated as follows. Let the *red degree* (*blue degree*, resp.) of a vertex v be the number of red (blue, resp.) edges incident to v. Alice (Bob, resp.) *captures* a vertex v if its red (blue, resp.) degree is more than $\lfloor d(v)/2 \rfloor$. If $d(v)$ is even and the red degree of v equals the blue degree of v, then neither player captures v. The score of a player is the number of vertices they captured. Once the scores achieved by Alice and Bob have been computed, the *outcome* of the game, *i.e.*, whether one of the two players wins and by how much they win, can be determined. The game ends in a win for Alice (Bob, resp.) if Alice's score (Bob's score, resp.) is greater than Bob's score (Alice's score, resp.). If both players achieve the same score, then the game ends in a draw.

Note that, while the Edge-Balanced Index Game was introduced in [5], the author only suggests the game and its rules, but does not provide any results on it. Furthermore, to the best of our knowledge, the game has not been studied since. Thus, our paper is the first work exploring this game, and so, all our results in this paper are new ones.

As mentioned earlier, this game takes place in the more general context of scoring games, which are games in which opposing players aim at achieving a score larger than that of their opponents, with the notion of a score being a measure computed relative to the set of rules of the game. Scoring game theory was first introduced in two works in the 1950s: one by Milnor [11], and one by Hanner [8]. While new results on the topic have only appeared in the last 30 years, these two pioneering works were at the heart of the construction of both the economic and combinatorial game theories. In [9], the authors give a survey of the general frameworks of resolutions for certain families (also known as *universes*) of scoring games. Two of the more well-known universes are Milnor's universe [11] and Ettinger's universe [7]. To define whether a scoring game can be embedded in either of these two universes, we first have to give some definitions. A game has no zugzwang if each player

[2]In this work, we only focus on connected graphs, and thus, the connectivity requirement is omitted throughout.

always prefers making their move rather than skipping their turn. A game is a dicot if, in any round of the game, a player can move if and only if his opponent also could. The scoring games that can be embedded in Milnor's universe are non-zugzwang dicot games, while those that can be embedded in Ettinger's universe are simply dicot games (zugzwang positions are allowed). It is easy to see that the Edge-Balanced Index Game can be embedded in Milnor's universe. For more on scoring games, we refer the reader to [9] for a survey on the topic, which also includes a formalism to deal with scoring games through scoring game notation, and the universes as mentioned above. To finish, we would like to mention a series of scoring games [2, 4, 6, 10, 12] that were recently introduced, and which feature different types of mechanisms or rules of independent interest.

Following the definitions in [9], we can associate a parameter to the Edge-Balanced Index Game that entails the exact outcome of the game (when both players play optimally) from its value. For any graph G, let $s(G)$ be the difference between the score of Alice and the score of Bob at the end of the Edge-Balanced Index Game in G (*i.e.*, $s(G) =$ score of Alice $-$ score of Bob), when Alice aims to maximise this difference and Bob aims to minimise it. In particular, whenever $s(G) > 0$ ($s(G) < 0$, resp.), Alice (Bob, resp.) has a winning strategy in G, and whenever $s(G) = 0$, the outcome of the game is a draw (when both players play optimally).

This paper is organised as follows. We start in Section 2 by exploiting classic strategy stealing arguments to show that $s(G) \in \{0, 1, 2\}$ for any graph G. In Section 3, we define two general classes of graphs for which $s(G) = 0$ for any graph G in either of these classes, and we investigate slight variations of these classes of graphs. Notably, through the study of these two classes of graphs, we then exhibit more mechanisms and subtleties of the game in Section 4, such as the role of vertices with given degree parity, and whether the parity of the size of a graph (*i.e.*, its number of edges) always swings the balance in favour of one of the players. We then focus on more classical classes of graphs through Sections 5 to 7, and, in particular, determine the exact outcome of the game in classes such as paths, cycles, complete bipartite graphs, and Cartesian grids. For trees and complete graphs, we provide partial results. In Section 8, we finish with a discussion featuring three interesting open questions.

2. Stealing strategies: The possible outcomes for the game

In this section, we deduce the possible outcomes for the Edge-Balanced Index Game through classic strategy-stealing arguments, which are based on a player stealing their opponent's strategy. This is common for impartial games in which playing an extra turn is never harmful for a player. In particular, a *strategy* $\mathcal{S}$ for a player is a map that takes the current state of the game as input, and outputs the next move for the player. In the context of the Edge-Balanced Index Game, the current state of a game played on a graph G is defined by three sets $R \subseteq E(G)$, $B \subseteq E(G)$, and $U \subseteq E(G)$, where R is the set of red edges, B is the set of blue edges, and U is the set of uncoloured edges. Thus, in the context of the Edge-Balanced Index Game, a strategy for a player in G takes R, B, and U as inputs, and outputs an edge of U to be coloured by that player. Furthermore, a *winning* strategy for a player is one that guarantees that player wins at the end of the game, regardless of how the other player plays. Lastly, for any $x \in \mathbb{N}$, note that there exists a strategy $\mathcal{S}$ in G for Alice that guarantees that her score is at least x larger than Bob's score if and only if there exists a strategy $\mathcal{S}'$ in G for Bob that guarantees that his score is at least x larger than Alice's score if instead Bob is the first player and Alice is the second player. This is rather obvious since it only involves switching the colours red and blue, and so, in the proofs of the theorems in this section, we abuse notation and simply refer to $\mathcal{S}'$ as $\mathcal{S}$.

On the one hand, we prove that if Alice plays optimally, then she can never lose the game. On the other hand, we show that Bob always has a strategy to ensure that Alice's score is at most 2 larger than Bob's score. Thus, while Bob can never win the game, he can always prevent an overwhelming win by Alice. We start by showing that Alice can always avoid losing.

Theorem 2.1. *For any graph G, $s(G) \geq 0$.*

Proof. Assume the contrary, and let G be a graph in which Bob has a winning strategy $\mathcal{S}$. Consider the following strategy for Alice to play in G. During her first turn, she colours any edge uv. She now sees the rest of the game as a new game in G, with Bob acting as the first player and her acting as the second player. More precisely, Alice follows the strategy $\mathcal{S}$ under the assumption that, initially, $uv \in U$. There is also the caveat that if, at some turn, by $\mathcal{S}$ she is supposed to colour an edge xy that is already coloured (then it must be red, and the first time this is possible is when $xy = uv$), then she adds xy to R, and colours any other uncoloured edge wz of G instead (if one still exists, and otherwise, the game is over), but leaves wz in U. Hence, she can follow $\mathcal{S}$ in G exactly as the second player could follow $\mathcal{S}$ in G. Note that, once the game ends, Alice has coloured at least the edges that she was supposed to colour by $\mathcal{S}$, which is a winning strategy, and thus, she wins in G. Consequently, Bob cannot win in G, a contradiction. □

Through the next result, we show that there are only three possible outcomes to the game when both players play optimally.

Theorem 2.2. *For any graph G, $s(G) \in \{0, 1, 2\}$.*

Proof. By Theorem 2.1, $s(G) \geq 0$, and so, we just need to prove that $s(G) \leq 2$. Assume the contrary, and let G be a graph in which Alice has a winning strategy $\mathcal{S}$ guaranteeing her score is at least 3 larger than Bob's score. Consider the following strategy for Bob to play in G. Assume Alice colours an edge uv during her first turn. Bob then sees the rest of the game as a new game in G, with Bob acting as the first player and Alice acting as the second player. Bob follows the strategy $\mathcal{S}$ under the assumption that, initially, $uv \in U$. There is also the caveat that if, at some turn, by $\mathcal{S}$ he is supposed to colour an edge xy that is already coloured (the first time this is possible is when $xy = uv$), then he adds xy to B (even though it may be red), and colours any other uncoloured edge wz of G instead (if one still exists, and otherwise, the game is over), but leaves wz in U. Hence, he can follow $\mathcal{S}$ in G exactly as the first player could follow $\mathcal{S}$ in G, except that there may be an edge that he should have coloured blue, that is in fact red. Indeed, once the game ends, Bob has coloured all the edges that he was supposed to colour by $\mathcal{S}$, except maybe uv. Note that there can be at most one such edge and it must be uv if it exists, since Alice only played one turn before Bob saw the rest of the game as a new game in G, and this was the first turn. Let us now analyse the score of Bob, which depends on whether he needed to colour uv as part of $\mathcal{S}$ or not.

- Assume first that Bob was able to follow $\mathcal{S}$ from start to end, *i.e.*, colouring uv was not part of $\mathcal{S}$. Then, Bob achieves a score that is at least 3 larger than Alice's score by the definition of $\mathcal{S}$.
- Assume now that Bob was supposed to colour uv at some point, but was actually not able to, since Alice coloured this edge during the first round. So uv is part of

the red subgraph, but we know that if this edge is moved to the blue subgraph, then Bob's score is at least 3 larger than Alice's score. Let us study the effect of having uv being in the red subgraph, and not in the blue subgraph.

Consider, say, u. Note that moving the edge uv from the blue subgraph to the red subgraph modifies the difference between the blue degree and red degree of u by exactly 2 (the blue degree decreases by 1, while the red degree increases by 1). From this, we deduce, upon having uv in the red subgraph and not in the blue subgraph as indicated by $\mathcal{S}$, the following:

- If, by $\mathcal{S}$, the second player was supposed to capture u, then Alice eventually captures u as intended. This does not alter the eventual score of either player.
- If, by $\mathcal{S}$, none of the players were supposed to capture u, then Alice eventually captures u. Then, Alice's eventual score is actually 1 larger than what it was supposed to be, while Bob's eventual score is not altered.
- If, by $\mathcal{S}$, the first player was supposed to capture u, then there are three cases to analyse:
  * If, by $\mathcal{S}$, the blue degree of u was supposed to be at least 3 larger than its red degree, then Bob eventually captures u as intended. This does not alter the eventual score of either player.
  * If, by $\mathcal{S}$, the blue degree of u was supposed to be exactly 2 larger than its red degree, then neither of the players eventually captures u. Then, Bob's eventual score is actually 1 smaller than what it was supposed to be, while Alice's eventual score is not altered.
  * If, by $\mathcal{S}$, the blue degree of u was supposed to be exactly 1 larger than its red degree, then Alice eventually captures u. Then, Bob's eventual score is actually 1 smaller than what it was supposed to be, while Alice's eventual score is actually 1 larger than it was supposed to be.

These arguments apply for both u and v, which implies that, in the worst-case scenario (*i.e.*, the scenario that changes the score the most in favour of Alice), Bob's score is 2 smaller than the score he was supposed to achieve through following $\mathcal{S}$, and Alice's score is 2 larger. More precisely, this corresponds to the situation where Bob was supposed to capture both u and v, while these vertices are actually captured by Alice due to her having coloured the edge uv that Bob was supposed to colour by $\mathcal{S}$.

Thus, overall, by following the strategy above, Bob guarantees that if Alice wins in G, then she wins with a score of at most 1 larger than his score. This contradicts the fact that Alice has a winning strategy in G ensuring her a score of at least 3 larger than Bob's score. □

The ideas from Theorem 2.2 actually have another interesting consequence in terms of winning strategies for Alice in which her score is at least 2 larger than Bob's score. In particular, the following theorem implies that, for any graph G such that $s(G) = 2$, G must contain an edge pq such that both p and q have odd degree.

Theorem 2.3. *For any graph G such that $s(G) = 2$, and any winning strategy $\mathcal{S}$ for Alice in G in which her score is at least 2 larger than Bob's score, in some round, she must colour an edge $pq \in E(G)$ such that p and q both have odd degree.*

Proof. Assume the contrary, and let G be a graph in which Alice has a winning strategy $\mathcal{S}$ guaranteeing her score is at least 2 larger than Bob's score, and, by $\mathcal{S}$, she never colours an edge $pq \in E(G)$ such that both p and q have odd degree. Consider the following strategy for Bob to play in G. Assume Alice colours an edge uv during her first turn. Bob then sees the rest of the game as a new game in G, with Bob acting as the first player and Alice acting as the second player. Just as in the proof of Theorem 2.2, Bob follows the strategy $\mathcal{S}$ under the assumption that, initially, $uv \in U$. There is also the caveat that if, at some turn, by $\mathcal{S}$ he is supposed to colour an edge xy that is already coloured (the first time this is possible is when $xy = uv$), then he adds xy to B (even though it may be red), and colours any other uncoloured edge wz of G instead (if one still exists, and otherwise, the game is over), but leaves wz in U. Hence, he can follow $\mathcal{S}$ in G exactly as the first player could follow $\mathcal{S}$ in G, except that there may be an edge that he should have coloured blue, that is in fact red. Indeed, once the game ends, Bob has coloured all the edges that he was supposed to colour by $\mathcal{S}$, except maybe uv. Note that there can be at most one such edge and it must be uv if it exists, since Alice only played one turn before Bob saw the rest of the game as a new game in G, and this was the first turn. Let us now analyse the score of Bob, which depends on whether he needed to colour uv as part of $\mathcal{S}$ or not.

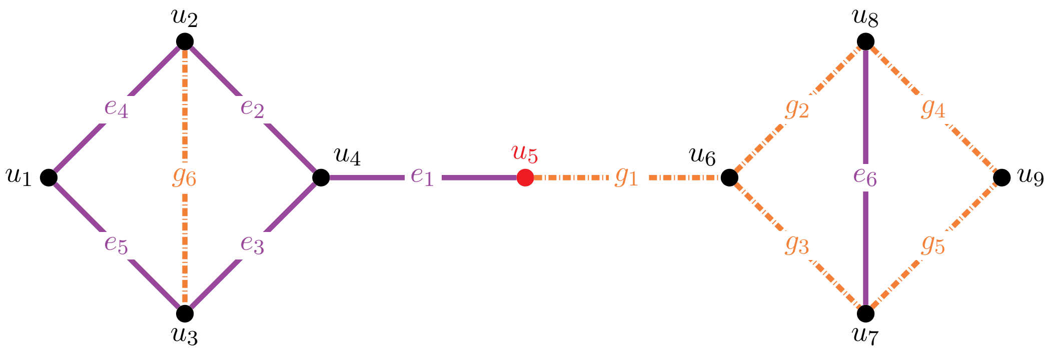
- Assume first that Bob was able to follow $\mathcal{S}$ from start to end, *i.e.*, colouring uv was not part of $\mathcal{S}$. Then, Bob achieves a score that is at least 2 larger than Alice's score, and so, we have a contradiction.

- Assume now that Bob was supposed to colour uv at some point, but was actually not able to, since Alice coloured this edge during the first round. So uv is part of the red subgraph, but we know that if this edge is moved to the blue subgraph, then Bob's score is at least 2 larger than Alice's score.

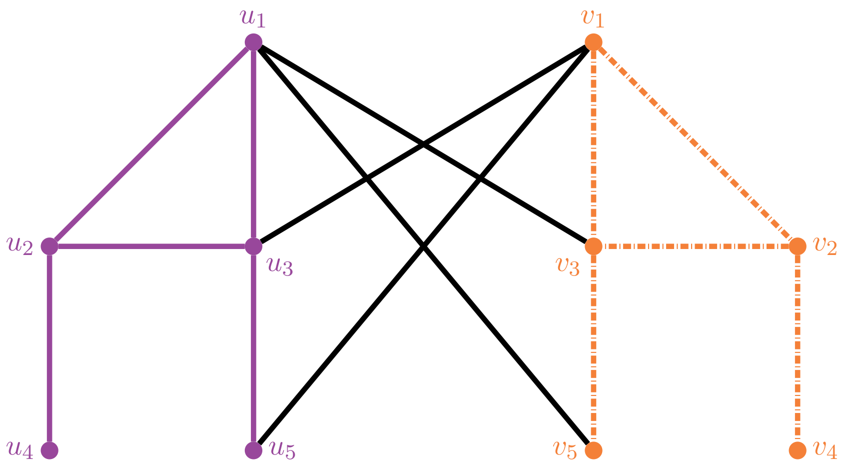
  Since at least one of u and v has even degree, say u, it is not possible, by $\mathcal{S}$, for the blue degree of u to be exactly 1 larger than its red degree, as this would imply it has odd degree. As was seen in the proof of Theorem 2.2, as long as, by $\mathcal{S}$, the blue degrees of both u and v were not supposed to be exactly 1 larger than their red degrees, then Alice's eventual score is at most 1 larger than Bob's eventual score. Thus, we have a contradiction. □

3. Splitting and folding graphs: Playing in symmetric graphs

In the next two subsections, we introduce two classes of graphs, that we call **splittable graphs** and **foldable graphs** (illustrated in Figure 1), which are graphs with a symmetrical structure, allowing Bob to copy Alice's strategies to force a draw. The main difference between these two types of structures, lies in that the symmetries of splittable graphs are with respect to their edges, while those of foldable graphs are with respect to their vertices. In particular, foldable graphs are a subclass of splittable graphs, however, they are simpler to understand and visualise, so we define and use them for this purpose. Note, however, that not every splittable graph is a foldable graph. In particular, all foldable graphs are of even order, while splittable graphs can be of odd order. We also exhibit classes of graphs that are very close to being splittable or foldable, for which the players can exploit the structure to reach a particular outcome for the game. One main point of interest for our results in this section, is that they apply to common classes of graphs, which are actually splittable or foldable (see, in particular, Section 5).



(a) A splittable graph G with parts $E_1 = (e_1, \dots, e_6)$ and $E_2 = (g_1, \dots, g_6)$. The subgraphs $G[E_1]$ (in solid purple edges) and $G[E_2]$ (in dashed orange edges) are isomorphic, and the function $f : V(G[E_1]) \to V(G[E_2])$ defined as $f(u_i) = u_{9-i+1}$ for every $i \in \{1, 2, 3, 4, 5, 7, 8\}$, is an isomorphism between $G[E_1]$ and $G[E_2]$. Note that, by f, every edge e_i gets mapped to the edge g_i. Note also that $f(u_5) = u_5$, and so, u_5 is a center vertex, while all the other u_i's are corner vertices.



(b) A foldable graph G with parts $U = (u_1, \dots, u_5)$ and $V = (v_1, \dots, v_5)$. The subgraphs $G[U]$ (in solid purple vertices and edges) and $G[V]$ (in dashed orange vertices and edges) are isomorphic, and the function $f : U \to V$ defined as $f(u_i) = v_i$ for every $i \in \{1, \dots, 5\}$, is an isomorphism between $G[U]$ and $G[V]$. No edge of the form $u_i v_i$ exists, while, for every edge of the form $u_i v_j$, we also have the edge $u_j v_i$.

Figure 1: Examples of splittable and foldable graphs.

3.1. Splittable graphs

A graph G is *splittable* if its edge set $E(G)$ can be partitioned into two ordered parts[3] $E_1 = (e_1, \dots, e_k)$ and $E_2 = (g_1, \dots, g_k)$ with the same cardinality, such that two properties hold. The first one of these properties is the following:

- $G_1 = G[E_1]$ and $G_2 = G[E_2]$ are isomorphic, and there exists an isomorphism $f : V(G_1) \to V(G_2)$ such that e_i gets mapped to g_i for every $i \in \{1, \dots, k\}$. That is, if $e_i = uv$ and $g_i = xy$, then $f(u) \in \{x, y\}$ and $f(v) \in \{x, y\} \setminus \{f(u)\}$.

The second property of splittable graphs deals with the correspondance between the

[3]Throughout the paper, to avoid stating the isomorphism between $G[E_1]$ and $G[E_2]$ each time, we instead use this ordering to describe the implicit isomorphism between them, *i.e.*, by the isomorphism $f : V(G_1) \to V(G_2)$, the i^{th} edge in the first ordered part always gets mapped to the i^{th} edge in the second ordered part.

vertices of G_1 and G_2 through the said isomorphism f. Namely:

- for every vertex $v \in V(G_1)$, either 1) $f(v) = v$, or 2) $f(v) = u$ and $f^{-1}(u) = v$ for some $u \in V(G_2)$.

Note that this last property yields a pairing of the vertices of G, where a pair consists of two (possibly identical) vertices being images of each other in G_1 and G_2 (through f). It is possible that a vertex might be paired to itself. Whenever dealing with a splittable graph G, for every vertex x of G, for legibility we denote by $f(x)$ the image (or preimage) of x by the isomorphism f mentioned in the definition. So we have either $f(x) = y$ and $f(y) = x$ for some $y \neq x$ (*corner vertices*), or $f(x) = x$ (*center vertex*). Two corner vertices x and y are *opposite* if $f(x) = y$ and $f(y) = x$. For an edge xy, we denote by $f(xy)$ the image (or preimage) of xy by f.

Note that if v is a vertex of G that is a center vertex, then v essentially plays the same role in G_1 and G_2. Since these subgraphs G_1 and G_2 are isomorphic, this implies that the number of edges of E_1 incident to v is equal to the number of edges of E_2 incident to v. In particular, any center vertex of G must be of even degree.

We prove that Bob can always ensure a draw in a splittable graph.

Theorem 3.1. *If G is a splittable graph, then $s(G) = 0$.*

Proof. Consider a game in G, and the strategy for Bob where, at each turn, he answers to Alice colouring an edge xy by colouring $f(xy)$. The definition of splittable graphs implies that, in each round, the edges that Alice and Bob colour are in two different edge-disjoint subgraphs with convenient intersection properties. In particular, whenever Alice colours an edge in one of these two subgraphs, then Bob is essentially, through a colouring, mimicking the play in the second subgraph. It is easy to see then, that once the game ends, the red and blue subgraphs are isomorphic, in such a way that Alice captures a vertex of the red subgraph if and only if Bob captures its image in the blue subgraph. Hence, the game ends in a draw. □

We now prove that modifying the structure of a splittable graph can have different consequences on the outcome of the game, and, in particular, make it lose the drawing property in Theorem 3.1, in a more or less strong way.

Observation 3.2. *Let H be a splittable graph with two non-adjacent center vertices u and v, and let G be the graph obtained from H by adding the edge uv. Then, $s(G) = 2$.*

Proof. Consider the following strategy for Alice. During the first turn, she colours uv. We then consider the game as a new game on H, with Bob acting as the first player and Alice acting as the second player. From now on, Alice reacts to Bob's moves according to the drawing strategy described in Theorem 3.1. As a result, once the game ends, what results is a draw in H, and, in particular, because u and v are center vertices, neither Alice nor Bob captures any of these two vertices (since any center vertex has the same number of incident edges in E_1 and E_2). Due to Alice having coloured uv in the first round, in G, she actually captures both u and v. The game thus ends in Alice winning with a score that is 2 larger than Bob's score. □

The previous observation shows a peculiar general property of the game, which is that altering the structure of a graph even slightly, for instance through the addition of just one edge, may have drastic effects on the outcome for the two players. That is, there are graphs G such that $s(G) = 0$, but for the graph G' obtained by adding one edge to G, it holds that $s(G') = 2$.

For the next result, we need an additional definition. Once a game in a graph ends, we say that a vertex is *barely captured* by Alice (Bob, resp.), if its red degree (blue degree, resp.) is 1 more than its blue degree (red degree, resp.).

Observation 3.3. *Let H be a splittable graph with two non-adjacent opposite corner vertices u and v, and let G be the graph obtained from H by adding the edge uv. If Bob has a drawing strategy in H for which u and v are barely captured by different players, then $s(G) = 1$.*

Proof. We first prove that $s(G) \geq 1$ by describing a strategy for Alice. Alice starts by colouring uv, and then, as the second player in H, she follows the drawing strategy in H for which u and v are barely captured by different players. Since u and v are barely captured by the players when omitting uv, when taking into account that uv was coloured by Alice, we get that Alice still captures one of u and v, while none of the players captures the second vertex. Then Alice's score is 1 larger than Bob's score and $s(G) \geq 1$.

The fact that $s(G) < 2$ follows from the fact that Bob has a strategy to prevent Alice from winning with a score that is at least 2 larger than his score. This strategy is as follows:

- If Alice colours uv, then Bob colours any other edge of G.
- Otherwise, Alice colours an edge of H, and then Bob colours an edge according to the drawing strategy in H. If that edge is already coloured, then Bob colours any other edge of H.

Note that H has an even number of edges as it is a splittable graph, and thus, G has an odd number of edges. Hence, through this strategy, the edge uv must be coloured by Alice. From this, it can be noted that Alice and Bob achieve a draw in H, due to how Bob reacted to Alice's moves. Furthermore, still in H, the ends of uv are barely captured by both players. The fact that Alice coloured uv implies that, in G, Bob eventually captures none of uv's ends. Thus, Alice achieves a score that is precisely 1 larger than Bob's score, and $s(G) = 1$. □

3.2. Foldable graphs

A graph G is *foldable* if its vertex set $V(G)$ can be partitioned into two ordered parts[4] $U = (u_1, \ldots, u_k)$ and $V = (v_1, \ldots, v_k)$ with the same cardinality, such that:

- the vertex-mapping $f : U \to V$, where $f(u_i) = v_i$ for every $i \in \{1, \ldots, k\}$, is an isomorphism between $G[U]$ and $G[V]$;
- for any two distinct $i, j \in \{1, \ldots, k\}$, if $u_i v_j \in E(G)$, then $u_j v_i \in E(G)$;
- for every $i \in \{1, \ldots, k\}$, the edge $u_i v_i$ does not exist.

Note that this definition implies that every foldable graph has even order and even size. Whenever dealing with a foldable graph G, for every vertex x of G, for legibility we denote

[4]Throughout the paper, to avoid stating the isomorphism between $G[U]$ and $G[V]$ each time, we instead use this ordering to describe the isomorphism between them, *i.e.*, by the isomorphism $f : U \to V$, the i^{th} element in the second ordered part is always the image of the i^{th} element in the first ordered part.

by $f(x)$ the image (or preimage) of x by the isomorphism f mentioned in the definition. For any edge xy of G, we denote by $f(xy)$ the image (or preimage) of xy.

As mentioned earlier, foldable graphs are always splittable. Indeed, assume G is a graph that is foldable, according to the terminology above. Consider the bipartition $E_1 \cup E_2$ of $E(G)$, where E_1 contains all the edges of $G[U]$, E_2 contains all the edges of $G[V]$, and, for every edge u_iv_j with $u_i \in U$ and $v_j \in V$, we add one of u_iv_j and u_jv_i to E_1, and the other edge to E_2. Then, it can be noted that E_1 and E_2 show that G is splittable. In particular, we have $f(u_i) = v_i$ for every $i \in \{1, \dots, k\}$. Also, since G is foldable, we have an even number of vertices in G. Note that it is possible, however, for splittable graphs to have odd order. This shows that splittable graphs are, in general, not foldable.

Since all foldable graphs are splittable graphs, Bob can always force a draw in a foldable graph. We prove this again, however, to give the explicit strategy for Bob.

Theorem 3.4. *If G is a foldable graph, then $s(G) = 0$.*

Proof. Assume Alice and Bob play in G, and consider the strategy for Bob, where, at every turn, he reacts to Alice colouring an edge xy by colouring $f(xy)$. Note that Bob can always play this way, regardless of the edge Alice colours, since it cannot be that the edge to colour in response is already coloured. Actually, through this strategy, the edges of G get coloured in pairs, in the sense that any two edges xy and $f(xy)$ are always coloured by Alice and Bob within a same round. To see now that the game ends in a draw, it is sufficient to note that the eventual red and blue subgraphs are isomorphic, and, in particular, the vertex-mapping f yields an isomorphism between the red and blue subgraphs. This implies that, for all $x \in V(G)$, if Alice (Bob, resp.) captures x, then Bob (Alice, resp.) captures $f(x)$. Furthermore, for all $x \in V(G)$, if none of the players captures x, then none of the players captures $f(x)$. Thus, Alice and Bob achieve the same score. □

In the next two results, we show that slightly tweaking the structure of a foldable graph can have different effects, such as maintaining that, for the new graph G, $s(G) = 0$, or making it so that $s(G) = 2$.

Observation 3.5. *Let H be a foldable graph, and let G be any graph obtained from H by adding a new vertex w, and adding pairs of edges of the form $\{wx, wf(x)\}$, where $x \in V(H)$. Then, $s(G) = 0$.*

Proof. Let us consider a game in G, and, in particular, the following strategy for Bob:

- If Alice colours an edge xy of H, then Bob colours $f(xy)$.
- If Alice colours an edge wx where $x \in V(H)$, then Bob colours $wf(x)$.

It can be checked that, once the game ends, the red and blue subgraphs are isomorphic, in such a way that Alice captures some x in $V(H)$ if and only if Bob captures $f(x)$. Also, note that the strategy above ensures that neither Alice nor Bob captures w. Thus, the game ends with Alice and Bob achieving the same score, thus in a draw. □

Observation 3.6. *Let H be a foldable graph, and let G be any graph obtained from H by adding two new vertices w_1 and w_2 joined by an edge, and adding pairs of edges of the form $\{w_1x, w_1f(x)\}$ and $\{w_2x, w_2f(x)\}$, where $x \in V(H)$. Then, $s(G) = 2$.*

Proof. Let us consider a game in G, and, in particular, the following strategy for Alice:

- During the first turn, Alice colours w_1w_2.

- From this point on, we see the rest of the game as a new game played on $G - w_1w_2$ with Bob acting as the first player each turn, and Alice playing as the second player. Alice's strategy from now on, is then answering to Bob's moves as follows:
  - If Bob colours an edge xy of H, then Alice colours $f(xy)$.
  - If Bob colours an edge w_1x (w_2x, resp.) where $x \in V(H)$, then Alice colours $w_1f(x)$ ($w_2f(x)$, resp.).

This strategy guarantees a draw in $G - w_1w_2$, as, in this graph, Bob captures a vertex x if and only if Alice captures $f(x)$, while w_1 and w_2 are captured by none of the players. Thus, due to Alice colouring w_1w_2 during the first turn, in G, the vertices w_1 and w_2 are actually both captured by Alice, while the situation remains unchanged for the other vertices. Overall, the eventual score of Alice is thus 2 more than that of Bob. □

4. Peculiar behaviours of the game

In this section, we exhibit peculiar behaviours of the game that depend on the parities of the degrees of the vertices of a graph, and we investigate the role the parity of the size of a graph plays in the outcome of the game. We have already seen an interesting property of the game regarding vertices with distinct degree parity in Section 2. In particular, recall that Theorem 2.3 implies that, for any graph G, if G does not contain an edge uv such that u and v both have odd degree, then $s(G) < 2$. Thus, from Theorem 2.3, we get the following corollary:

Corollary 4.1. *If G is a graph such that all its vertices have even degree, then $s(G) \in \{0, 1\}$.*

A result of a similar flavour can be obtained for graphs containing only vertices of odd degree. Indeed, note that, while, on the one hand, even-degree vertices can be captured by either of the players or by none of them, odd-degree vertices, on the other hand, always end up captured by one of the two players. From this, we get the following result akin to that of Corollary 4.1:

Observation 4.2. *If G is a graph such that all its vertices have odd degree, then $s(G) \in \{0, 2\}$.*

Proof. Recall that, in every graph, the number of vertices with odd degree must be even. Thus, because all vertices of G have odd degree, we deduce that G has even order. Consider now the outcome of a game in G. Since a vertex with odd degree always ends up captured by one of the two players, we get that the number of vertices captured by Alice and the number of vertices captured by Bob have the same parity. Thus, the difference between the scores achieved by the two players must be even. From this, and the fact that $s(G) \in \{0, 1, 2\}$ (by Theorem 2.2), we deduce that $s(G) \neq 1$. Thus, $s(G) \in \{0, 2\}$. □

It is worth mentioning that both situations in both Corollary 4.1 and Observation 4.2 are plausible in general. That is, there exist infinitely many graphs G with no odd-degree vertices such that $s(G) = 0$ ($s(G) = 1$, resp.), and infinitely many graphs G with no even-degree vertices such that $s(G) = 0$ ($s(G) = 2$, resp.). We prove this through the following three results, with the first one holding even for regular graphs.

Corollary 4.3. *For any odd integer $k \geq 3$, there are arbitrarily large k-regular graphs G such that $s(G) = 0$.*

Proof. Consider the following construction. Let H_1 and H_2 be two copies of a $(k-1)$-regular graph. Denote by $U = (u_1, \dots, u_n)$ and $V = (v_1, \dots, v_n)$ the vertices of H_1 and H_2, respectively, such that the vertex-mapping $f : U \to V$, where $f(u_i) = v_i$ for all $i \in \{1, \dots, n\}$, is an isomorphism between H_1 and H_2. For each $i \in \{1, \dots, n\}$, add the edge $u_i v_{n+1-i}$, and call this resulting graph G. It is clear that G is k-regular, and $s(G) = 0$ by Theorem 3.4 since it is foldable. □

Corollary 4.4. *There are arbitrarily large graphs G with no even-degree vertices such that $s(G) = 2$.*

Proof. Consider the following construction. Let H_1 and H_2 be two copies of a k-regular graph for any odd integer $k \geq 3$. Denote by $U = (u_1, \dots, u_n)$ and $V = (v_1, \dots, v_n)$ the vertices of H_1 and H_2, respectively, such that the vertex-mapping $f : U \to V$, where $f(u_i) = v_i$ for all $i \in \{1, \dots, n\}$, is an isomorphism between H_1 and H_2. Add two vertices w_1 and w_2 joined by an edge, and, for each $i \in \{1, \dots, n\}$, add the edges $w_1 u_i$, $w_1 v_i$, $w_2 u_i$, $w_2 v_i$, and call this resulting graph G. Since k is odd, there are no vertices of even degree, and $s(G) = 2$ by Observation 3.6. □

Corollary 4.5. *There are arbitrarily large graphs G with no odd-degree vertices such that $s(G) = 0$ ($s(G) = 1$, resp.).*

Proof. Arbitrarily large such graphs can be constructed through exploiting the structure of foldable and splittable graphs, so that, for instance, Theorem 3.4 or Observation 3.3 applies. In particular, as will be seen later, by Theorem 5.2, $s(C_{2n+1}) = 1$ for every cycle C_{2n+1} of odd length, and $s(C_{2n}) = 0$ for every cycle C_{2n} of even length, while, in both cases, the degree condition of the statement is verified. □

Further more specific questions for graphs with only vertices of even degree can be asked. For instance, are there graphs G with $s(G) = 0$, where the only way for there to be a draw is for both players to have a score of 0 at the end, *i.e.*, none of the vertices are captured by the players? Easy arguments show that such graphs do not exist.

Observation 4.6. *There is no graph G with no odd-degree vertices such that $s(G) = 0$ and all games ending in a draw have no vertex being captured.*

Proof. Consider a game, played on a graph G with no odd-degree vertices, that ends in a draw such that no vertex is captured by the players. W.l.o.g., we assume that Alice was the second-to-last player to colour an edge e, while Bob was the last player to colour an edge f (*i.e.*, G has even size). We claim that the similar game played on G, but with Alice colouring f and Bob colouring e during their last turns, ends in a draw with some vertices being captured.

Assume first that $e = uv$ and $f = wx$ are disjoint. In the original game, the fact that none of u, v, w, and x get captured by the end of the game, means that, prior to the last round, the blue degrees of u and v are 1 larger than their red degrees, and the red degrees of w and x are 1 larger than their blue degrees. Thus, we deduce that the modified game ends up with u and v being captured by Bob, and w and x being captured by Alice. The two players thus achieve the same score, and u, v, w, and x get captured.

Now, if, say, $e = uv$ and $f = vw$, *i.e.*, e and f share an end v, then prior to the last round in the original game, the blue degree of u is 1 larger than its red degree, the red degree of w is 1 larger than its blue degree, and v has the same red degree and blue degree. Here, the modified game ends with u being captured by Bob, w being captured by Alice, and v being captured by neither of the players. Thus, we again get a draw, but with u and w being captured. □

	Odd size	**Even size**
$\mathbf{s(G) = 0}$	Particular graphs (Thm. 7.7)	Splittable graphs (Thm. 3.1) Foldable graphs (Thm. 3.4)
$\mathbf{s(G) = 1}$	P_n, $n > 2$ even (Thm. 5.1) C_n, $n \geq 3$ odd (Thm. 5.2) $G_{n,m}$, $2 < n < m$, $n \not\equiv m \bmod 2$ (Thm. 5.4)	Particular trees (Thm. 6.5)
$\mathbf{s(G) = 2}$	$K_{n,m}$, $n, m \geq 2$ odd (Thm 5.3) $G_{2,n}$, $n \geq 3$ odd (Thm 5.4)	Particular graphs (Thm. 6.6)

Table 1: Examples of arbitrarily large graphs with given size and outcome for the game.

Another interesting question to ask, is whether Alice has a distinct advantage in graphs with odd size, since Alice might seem favoured due to her starting the game and getting to colour one more edge than Bob. Surprisingly enough, we show that, for every $x \in \{0, 1, 2\}$, there exist arbitrarily large graphs G of odd (even, resp.) size with $s(G) = x$. While this can be shown true for some combinations by studying common classes of graphs, for others, more artificial examples of graphs are needed.

Corollary 4.7. *For every $x \in \{0, 1, 2\}$, there exist arbitrarily large graphs G of:*

- *odd size with $s(G) = x$;*
- *even size with $s(G) = x$.*

Proof. The claim follows from the results to be established in the next sections. Table 1 provides a summary of possible graph classes illustrating each case. □

It is worth mentioning that the classes of graphs mentioned in Table 1 form an illustrative sample only. In particular, as is going to be seen later through Lemma 6.4, there exist graph transformations that can be used to construct bigger and bigger graphs, while preserving both the size parity and the outcome of the game.

5. Outcome of the game in common graph classes

Employing the tools introduced in Section 3, we determine the outcome of the game in common classes of graphs, including paths and cycles, complete bipartite graphs, and Cartesian grids. Precisely, for each graph G in those classes, we determine $s(G)$.

For any $n \geq 2$, we denote by P_n the path of order n. For any $n \geq 3$, we denote by C_n the cycle of order n. For any two $n, m \geq 1$, we denote by $K_{n,m}$ the complete bipartite graph in which the two partite sets have cardinality n and m, respectively. For any two $n, m \geq 2$, we denote by $G_{n,m}$ the Cartesian grid with n rows and m columns (*i.e.*, the Cartesian product of P_n and P_m).

Theorem 5.1. *Let $n \geq 2$. Then,*

$$s(P_n) = \begin{cases} 2 & \text{if } n = 2, \\ 1 & \text{if } n \geq 4 \text{ and } n \text{ is even}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $P = P_n$ for some $n \geq 2$. We denote by $v_1, \dots, v_n$ the consecutive vertices of P. If $n = 2$, then it is clear that $s(P) = 2$. If n is odd, then $s(P) = 0$ by Theorem 3.1 since it is splittable, as can be seen by the bipartition of $E(P)$ into the two parts $(v_1v_2, \dots, v_{\lfloor n/2 \rfloor}v_{\lceil n/2 \rceil})$ and $(v_nv_{n-1}, \dots, v_{(\lceil n/2 \rceil)+1}v_{\lceil n/2 \rceil})$. Now, if n is even and $n \geq 4$, then $s(P) = 1$ by Observation 3.3. Indeed, note that P can be seen as the splittable graph $P - v_{n/2}v_{(n/2)+1}$ with the edge bipartition with parts $(v_{n/2}v_{(n/2)-1}, \dots, v_2v_1)$ and $(v_{(n/2)+1}v_{(n/2)+2}, \dots, v_{n-1}v_n)$ (not joined by any edge), to which we have added the edge $v_{n/2}v_{(n/2)+1}$. In particular, note that, because $v_{n/2}$ and $v_{(n/2)+1}$ both have degree 1 in $P - v_{n/2}v_{(n/2)+1}$, the drawing strategy in splittable graphs guarantees that both $v_{n/2}$ and $v_{(n/2)+1}$ are barely captured by Alice and Bob. Thus, all the conditions are met for Observation 3.3 to apply. □

Theorem 5.2. *Let $n \geq 3$. Then,*

$$s(C_n) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $C = C_n$ for some $n \geq 3$. Let us denote by $v_1, \dots, v_n$ the consecutive vertices of C, where v_1v_n is an edge. First, observe that C is a foldable graph when n is even, and thus, $s(C) = 0$ in such cases. To see this is true, it suffices to observe that the bipartition of $V(C)$ as two ordered parts $(v_1, \dots, v_{n/2})$ and $(v_{(n/2)+1}, \dots, v_n)$ fulfils the folding property.

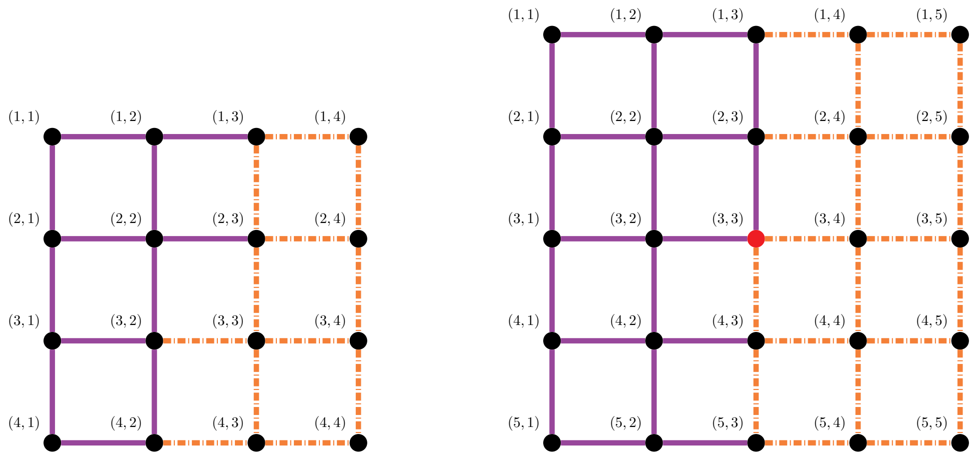
Now assume that n is odd. First note that C is edge-transitive, so we may assume, w.l.o.g., that Alice colours v_1v_2 in the first round of any game. From now on, we consider the rest of the game as a new game in $C' = C - v_1v_2$, a path of odd order, with Bob playing as the first player and Alice playing as the second player. Note that C' is a splittable graph, as noted in the proof of Theorem 5.1. Furthermore, it can be noted that the drawing strategy for the second player in an odd-order path, makes the two degree-1 vertices get captured by different players, and, due to their degrees being 1, being barely captured. Thus, Observation 3.3 applies, showing that $s(C) = 1$. □

Theorem 5.3. *Let $n, m \geq 1$. Then,*

$$s(K_{n,m}) = \begin{cases} 2 & \text{if } n \text{ and } m \text{ are odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $n, m \geq 1$ be fixed, and set $K = K_{n,m}$. Let us denote by (U, V) the bipartition of the vertices of K, where $|U| = n$ and $|V| = m$. We split the proof into three cases.

- Assume first that both n and m are even. In that case, we have $s(K) = 0$ directly from Theorem 3.4, since K is a foldable graph. To see this is true, label the vertices of U in an arbitrary way as $u_1, u'_1, \dots, u_{n/2}, u'_{n/2}$, and those in V as $v_1, v'_1, \dots, v_{m/2}, v'_{m/2}$, and note that K meets the definition of a foldable graph for the bipartition of its vertex set with parts $(u_1, \dots, u_{n/2}, v_1, \dots, v_{m/2})$ and $(u'_1, \dots, u'_{n/2}, v'_1, \dots, v'_{m/2})$.
- Assume now, w.l.o.g., that n is odd and m is even. If $n = 1$, then K is a star, and, regardless of how the players play, the game ends in a draw since m is even. So assume $n \geq 3$. Let w be any vertex of U, and set $K' = K - w$. As seen in the previous case, K' is a foldable graph. Since w is joined to every vertex of V, it is easy to see that K fulfils the conditions in the statement of Observation 3.5. Thus, there is a drawing strategy for Bob, and $s(K) = 0$.
- Lastly, assume that n and m are both odd. We can further assume that $n, m \geq 3$, as otherwise K would be a star with an odd number of leaves, in which case any game on K ends with Alice winning by 2. Let thus $w_1 \in U$ and $w_2 \in V$ be any two adjacent vertices of K, and set $K' = K - w_1 - w_2$. Here as well, K' is a foldable graph. Since



(a) $G_{4,4}$ (b) $G_{5,5}$

Figure 2: Cartesian grids with even size are splittable. One part of the edge bipartition contains the purple solid edges, while the second part contains the orange dashed edges. Vertices in red are center vertices.

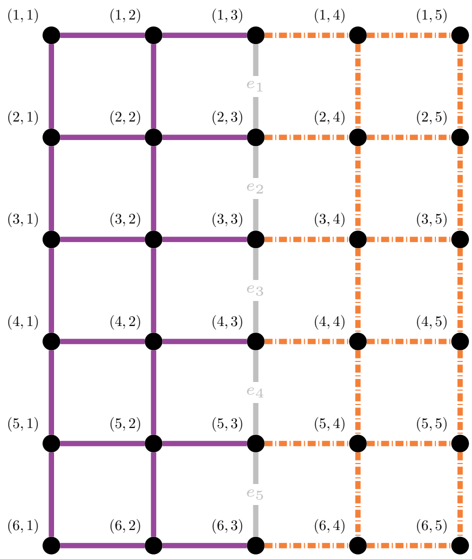


Figure 3: Accompanying illustration for the case of grids with an even number n of rows and an odd number m of columns in the proof of Theorem 5.4. When removing the edges $e_1, \ldots, e_{n-1}$, what remains is a splittable graph with the edge bipartition given by the purple solid edges and the orange dashed edges.

w_1 and w_2 are adjacent and joined to all the vertices of V and U, respectively, then K fulfils all the conditions in the statement of Observation 3.6, and thus, $s(K) = 2$. □

Theorem 5.4. *Let $n, m \geq 2$. Then,*

$$s(G_{n,m}) = \begin{cases} 2 & \text{if } n \text{ and } m \text{ have distinct parity and } 2 \in \{n, m\}, \\ 1 & \text{if } n \text{ and } m \text{ have distinct parity and } 2 \notin \{n, m\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $n, m \geq 2$ be fixed, and set $G = G_{n,m}$. For $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, we denote by (i, j) the vertex in row i and column j of G. We consider two cases:

- If n and m have the same parity, then G is actually a splittable graph, and the result follows from Theorem 3.1. To see this is true, consider the following partitions $E_1 \cup E_2$ of the edge set of G (see Figure 2 for an illustration):
  - If n and m are both even, then E_1 contains all of the edges from columns $1, \dots, m/2$, and all of the edges induced by their ends. E_1 also contains the edge $(i, m/2)(i, (m/2)+1)$ for every $i \in \{1, \dots, n/2\}$. All the other edges are in E_2. Note that, with respect to this partition of the edges, G is a splittable graph with no center vertex, and $f((i,j)) = (n-i+1, m-j+1)$ for every $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$.
  - If n and m are both odd, then E_1 contains all of the edges from columns $1, \dots, \lfloor m/2 \rfloor$, and all of the edges induced by their ends. E_1 also contains the edge $(i, \lfloor m/2 \rfloor)(i, \lfloor m/2 \rfloor + 1)$ for every $i \in \{1, \dots, n\}$. Finally, E_1 also contains the edge $(i, \lceil m/2 \rceil)(i+1, \lceil m/2 \rceil)$ for every $i \in \{1, \dots, \lfloor n/2 \rfloor\}$. All the other edges are in E_2. With respect to this partition of $E(G)$, note that G is a splittable graph with $f((i,j)) = (n-i+1, m-j+1)$ for every $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. In particular, $(\lceil n/2 \rceil, \lceil m/2 \rceil)$ is the unique center vertex.
- Now assume n and m have different parities, say, n is even and m is odd. Let $G' = G - \{e_1, \dots, e_{n-1}\}$, where, for legibility, for each $i \in \{1, \dots, n-1\}$, we set $e_i = (i, \lceil m/2 \rceil)(i+1, \lceil m/2 \rceil)$. Note that G' is splittable, as shown by the partition $E_1 \cup E_2$ of its edges, where E_1 contains the edges of columns $1, \dots, \lfloor m/2 \rfloor$ and all of the edges induced by their ends, as well as every edge $(i, \lceil m/2 \rceil - 1)(i, \lceil m/2 \rceil)$ with $i \in \{1, \dots, n\}$, while E_2 contains all the other edges. With respect to this partition of $E(G')$, note that G' is a splittable graph with $f((i,j)) = (i, m-j+1)$ for every $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. In particular, for each $i \in \{1, \dots, n\}$, the vertex $(i, \lceil m/2 \rceil)$ is a center vertex by f. See Figure 3 for an illustration.

  We consider the strategy for Alice in G where she plays as follows:

  - During the first round, Alice colours $e_{n/2}$.
  - From the second round onwards, Alice answers to Bob's moves as follows:
    * if $n > 2$ and Bob colours e_i for any $i \in \{1, \dots, n/2 - 1\}$, then Alice colours e_{n-i} and *vice versa*;
    * otherwise, *i.e.*, if Bob colours an edge h of G', then Alice colours the edge $f(h)$.

  As a result, we note that Alice and Bob achieve the same score in $G - e_{n/2}$ since G' is splittable and, for each $i \in \{1, \dots, n\}$, the vertex $(i, \lceil m/2 \rceil)$ is a center vertex by f, and thus, by the above strategy, if $n > 2$, then Alice and Bob will each capture the same number of these center vertices (by f) in $G - e_{n/2}$, and if $n = 2$, then neither of these two center vertices (by f) will be captured in $G - e_{n/2}$. In particular, when $n > 2$, the vertices $(n/2, \lceil m/2 \rceil)$ and $((n/2)+1, \lceil m/2 \rceil)$ get barely captured by different players. In G, the fact that Alice coloured $e_{n/2}$ during her first turn implies that one of these two vertices is captured by Alice while the other is captured by none of the players when $n > 2$, and that both these vertices get captured by Alice when $n = 2$. Thus, when $n = 2$, Alice wins with a score that is 2 larger than Bob's score, and when $n > 2$, Alice wins with a score that is 1 larger than Bob's score. Hence, we have shown that (when n is even and m is odd) $s(G) \geq 1$, and that $s(G) = 2$ when $n = 2$ (and m is odd).

To see that $s(G) < 2$ when $n > 2$ (for even n and odd m), consider the strategy where Bob reacts to Alice's moves as follows:

- If Alice colours $e_{n/2}$, then Bob colours any uncoloured edge of G.
- if Alice colours e_i for any $i \in \{1, \dots, n/2 - 1\}$, then Bob colours e_{n-i} and *vice versa*. If Bob already coloured that edge, then he colours any other edge of G.
- Otherwise, *i.e.*, Alice colours an edge h (of G'), then Bob colours the edge $f(h)$. If Bob already coloured that edge, then he colours any other edge of G.

As a result, similarly to as in the proof of Observation 3.3, through this strategy, the edge $e_{n/2}$ must be coloured by Alice. It can then be noted that Alice and Bob achieve a draw in $G - e_{n/2}$, due to how Bob reacted to Alice's moves. Furthermore, still in $G - e_{n/2}$, the vertices $(n/2, \lceil m/2 \rceil)$ and $((n/2) + 1, \lceil m/2 \rceil)$ (the ends of $e_{n/2}$ in G) are barely captured by both players. The fact that Alice coloured $e_{n/2}$ implies that, in G, Bob eventually captures none of $e_{n/2}$'s ends. Thus, Alice achieves a score that is precisely 1 larger than Bob's score, and $s(G) = 1$. □

6. Outcome of the game in trees

In this section, we study the game in trees, guided mainly by the upcoming conjecture. Recall that we have proved, in Corollary 4.7, that, contrarily to what one could think, graphs with odd size are not always the most favourable for Alice, while, on the contrary, Bob is not always guaranteed to prevent Alice from achieving the best possible score in a graph with even size. Looking closely at the graph classes we have provided as evidence in Table 1, it can be noted that this observation does not hold immediately when restricted to trees (some of the provided classes not being trees). Supported by computer experimentations (led on trees on up to 10 vertices), we actually suspect that trees might actually form a class of graphs in which the size is a crucial parameter. That is, we have the following conjecture:

Conjecture 6.1. *Let T be a tree. Then,*

- $s(T) \in \{1, 2\}$ *if T has odd size;*
- $s(T) \in \{0, 1\}$ *if T has even size.*

It can be noted that the classes of trees we have investigated in previous sections do not contradict Conjecture 6.1. In particular, the conjecture holds for paths (recall Theorem 5.1), while any tree T that is a splittable graph has even size, and so, $s(T) = 0$ by Theorem 3.1, and thus, the conjecture also holds for these trees.

The rest of this section is dedicated to introducing tools and approaches to progress towards understanding Conjecture 6.1. In particular, we are able to confirm this conjecture for several classes of trees, and we also prove that there actually exist infinitely many trees with the said properties. Some of the tools we introduce are also of more general interest. For instance, some of the constructions exhibited to prove Corollary 4.7, originate from our investigations in this section.

We start by introducing a new concept, motivated by the following ideas. The proofs of Theorems 3.1 and 3.4 essentially hold because, in a splittable or foldable graph, we can arrange the edges in pairs, so that a naïve drawing strategy for Bob is, at each turn, to colour the second edge in the pair that contains any edge Alice has just coloured. The success of this strategy is of course highly dependent on the graph's structure, and on how the pairs were formed. The next concept involves those ideas, leading to results in particular graph classes (including some classes of trees).

For a graph G with even size, we define a *pairing* over the edges of G as a collection $\mathcal{P}$ of pairs $\{e, f\}$ of (distinct) edges e and f, such that:

- $P_1 \cap P_2 = \emptyset$ for every two distinct $P_1, P_2 \in \mathcal{P}$,
- $\bigcup_{\{e,f\} \in \mathcal{P}} \{e, f\} = E(G)$.

Given a game on G, we define the *pairing strategy* (following $\mathcal{P}$) for Bob as the strategy where, at each turn, if Alice colours an edge e, then Bob colours the unique edge f such that $\{e, f\} \in \mathcal{P}$. This strategy is well defined, given that $\mathcal{P}$ fulfils the conditions above. It is worth adding that this notion of a pairing strategy for positional games is not a novel one, as it was featured in works dating back to the 1980s, such as that of Beck [1].

The next proof shows a situation in which pairing strategies come up naturally.

Lemma 6.2. *If T is a tree with a unique vertex of even degree, then $s(T) = 0$.*

Proof. Let r denote the unique vertex of even degree in T. We root T at r, thereby defining the usual root-to-leaves orientation of T, and the common notions of parent and children. Note that the condition on T implies its size is even. We define a pairing $\mathcal{P}$ over the edges of T, in the following way. Consider every vertex v with d children $u_1, \dots, u_d$ ($d \geq 0$). Since r is the unique vertex of even degree in T, note that d is even. Then we add, to $\mathcal{P}$, the pairs $\{vu_1, vu_2\}, \dots, \{vu_{d-1}, vu_d\}$.

We claim that sticking to the pairing strategy following $\mathcal{P}$, guarantees a draw for Bob. To see this is true, it suffices to note that, when doing so, for every vertex v with children $u_1, \dots, u_d$, Alice colours exactly $d/2$ of the edges $vu_1, \dots, vu_d$ while Bob colours the other $d/2$ edges. Thus, the status of whether v is captured by a player depends only on whether v has a parent w, and, in case it does, v is captured by the player that coloured wv. From these arguments, we deduce that the game ends up with r being captured by none of the players, while, for every non-leaf vertex v with d children, Alice and Bob both capture $d/2$ of these d children. Thus, Alice and Bob capture exactly the same number of vertices, and the game ends in a draw. □

We now turn our attention to trees with odd-degree vertices only, showing that Observation 4.2 can be refined further in this context.

Corollary 6.3. *If T is a tree in which all of its vertices have odd degree, then $s(T) = 2$.*

Proof. We may assume that T has two adjacent vertices r_1 and r_2 of odd degree at least 3, as, otherwise, T would be a star with an odd number of leaves, in which case the claim is easy to verify. Note that $T' = T - r_1r_2$ is a forest consisting of two trees T'_1 and T'_2 that both have only one vertex of even degree, r_1 and r_2, respectively. By Lemma 6.2, we have $s(T'_1) = s(T'_2) = 0$. Furthermore, as noted in the proof of that lemma, there is a drawing strategy for the second player in both T'_1 and T'_2 by which r_1 and r_2 do not get captured.

Consider now the following strategy for Alice in T:

- During the first turn, Alice colours r_1r_2.
- From this point on, Alice reacts to Bob's moves in T', as follows:
  - If Bob colours an edge in T_1, then Alice also colours an edge of T_1, following the drawing strategy ensuring r_1 will eventually be captured by none of the players.

– If Bob colours an edge in T_2, then Alice also colours an edge of T_2, following the drawing strategy ensuring r_2 will eventually be captured by none of the players.

Following this strategy, the game ends with the two players drawing in T', and with r_1 and r_2 being captured by none of the players in T'. Because r_1r_2 was coloured by Alice during the first turn, in T, the vertices r_1 and r_2 are actually captured by Alice, guaranteeing her a score that is 2 larger than Bob's score. □

Note that the two previous results agree with Conjecture 6.1, since a tree with only one vertex of even degree has even size, while a tree with only vertices of odd degree has odd size. Furthermore, note that Lemma 6.2 covers some well-studied classes of trees such as full binary trees (binary trees in which every non-leaf vertex has exactly two children).

One promising way to find examples of trees contradicting Conjecture 6.1, could be to study the structure of a minimum counterexample. This leads us to studying graph transformations that preserve the outcome of the game when performed on a given graph. In particular, the next result gives conditions under which removing particular structures from a graph preserves the outcome.

Lemma 6.4. *Let G be a graph, v be a vertex of G, and H be obtained from G by attaching, at v, two pending paths P and Q with lengths p and q, respectively, fulfilling one of the following conditions:*

1. $p = q = 1$,
2. $p, q \geq 2$ *are both even, or*
3. $p, q \geq 3$ *are both odd.*

If $s(G) = x$ for some $x \in \{0, 1, 2\}$, then $s(H) = x$.

Proof. The conditions on p and q imply that $p + q$ is even. We denote by $e_1, \dots, e_p$ and $f_1, \dots, f_q$ the consecutive edges of P and Q, respectively, where e_1 and f_1 are the only edges of P and Q incident to v. Let us now consider a pairing $\mathcal{P}$ over the edges of $E(P) \cup E(Q)$, built according to the condition p and q verify:

1. if $p = q = 1$, then $\mathcal{P} = \{\{e_1, f_1\}\}$;
2. in all the other cases, we add $\{e_1, f_1\}$ and $\{e_p, f_q\}$ to $\mathcal{P}$, and then we pair the other edges in $E(P) \cup E(Q)$ arbitrarily, and add the resulting pairs to $\mathcal{P}$.

Assume now that $s(G) = x$ for some $x \in \{0, 1, 2\}$. This means that Alice has a strategy $\mathcal{S}_A$ in G to end the game with an eventual score at least x larger than Bob's score, and that Bob has a strategy $\mathcal{S}_B$ to ensure Alice's score does not get more than x larger than his score. From $\mathcal{S}_A$ and $\mathcal{S}_B$, we derive strategies for Alice and Bob in H, showing that $s(H) = x$.

- Consider first the following strategy for Alice in H. She starts playing in G according to $\mathcal{S}_A$. If, at some point, Bob colours an edge e of P or Q, then she colours the unique edge f such that $\{e, f\} \in \mathcal{P}$, and then resumes her original strategy, (*i.e.*, reacts to where Bob plays next). In case all of the edges of G are coloured, but H still has uncoloured edges (of P and Q), then she colours any edge of P and Q, and then reacts to Bob's moves following $\mathcal{P}$ (in case she cannot, she, again, colours any remaining uncoloured edge of P or Q).

Once the game ends, it can be noted that the scores achieved in H by the two players, when only counting the captured vertices of G, are exactly the scores they would have achieved when playing the same way in G. This is because only the neighbourhood of v was altered when constructing H from G, and, by the strategy for Alice above, e_1 and f_1 are coloured by different players, implying that v remains captured by the same player that would have captured it in G with Alice following $\mathcal{S}_A$. Thus, when restricting the game on H to G, Alice achieves a score at least x larger than Bob's score.

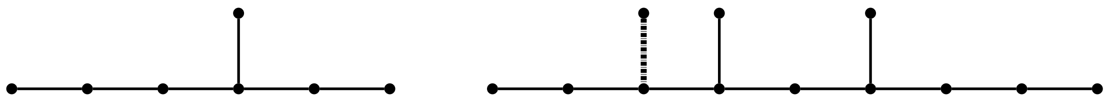
Now, we need to analyse how these scores are altered due to how the edges of P and Q were coloured by Alice and Bob. Note that P and Q induce a single path R of even length $p+q$. Also, due to how $\mathcal{P}$ was built and to the strategy for Alice above, Alice has coloured exactly half of the edges of R, and the end edges (e_p and f_q) of R have been coloured by different players. Note that the red and blue subgraphs of R have the same number of connected components (paths) since e_p and f_q have different colours by Alice's strategy. So, let us denote by $A_1, \dots, A_y$ and $B_1, \dots, B_y$ the connected components of the red and blue subgraphs, respectively, of R. For all $1 \leq i \leq y$, let a_i be the number of edges in the connected component A_i of the red subgraph of R, and let b_i be the number of edges in the connected component B_i of the blue subgraph of R. Note that, for any $1 \leq i \leq y$, the connected component of the red subgraph (blue subgraph, resp.) with size a_i (b_i, resp.) increases Alice's score (Bob's score, resp.) by a_i-1 (b_i-1, resp.) if it contains no ends of R, and otherwise, it increases Alice's score (Bob's score, resp.) by a_i (b_i, resp.). Since R is of even size and Alice has coloured exactly half of its edges, we have that $\sum_{i=1}^{y} a_i = \sum_{i=1}^{y} b_i$. Hence, since e_p and f_q have different colours by Alice's strategy, Alice and Bob achieve the same score in R. Thus, overall, Alice's score remains at least x larger than Bob's score in H.

- The strategy for Bob in H is similar to that for Alice above, except that, when reacting to Alice's moves in G, Bob follows his strategy $\mathcal{S}_B$. That is, Bob reacts to Alice playing in G by colouring, if possible, an edge of G according to $\mathcal{S}_B$. If no such edge of G remains, then Bob colours any edge of P and Q before reacting to Alice's moves in P and Q. To complete the strategy, Bob reacts to Alice colouring an edge e of P or Q by colouring the unique edge f such that $\{e, f\} \in \mathcal{P}$. Again, at any moment, if Bob is supposed to colour an edge that was already coloured, then he colours any remaining uncoloured edge of P or Q.

  Once the game ends, then, by the previous arguments, it can be checked that Alice and Bob achieve, when only counting the vertices captured in G, the same score they would have achieved with Bob following the strategy $\mathcal{S}_B$ in G. Thus, this far, Alice's score is at most x larger than Bob's score. Still by the arguments used above, the fact that the edges of P and Q were coloured in pairs, with e_p and f_q being coloured by different players, implies that Alice and Bob achieve the same score in R. Thus, following the strategy above, Alice's score remains at most x larger than Bob's score in H. □

Lemma 6.4 implies that, given a graph G with $s(G) = x$ for some $x \in \{0, 1, 2\}$, we can build infinitely many graphs H with $s(H) = x$. This has several consequences. For instance, we can prove that there exist infinitely many graphs G of even size with $s(G) = 1$, thus filling in one of the cells of Table 1.

Theorem 6.5. *There exist arbitrarily large graphs G of even size with $s(G) = 1$.*



(a) A tree T of even size with $s(T) = 1$

(b) A caterpillar T' with $s(T') = 2$

Figure 4: Two trees with particular properties.

Proof. This follows from Lemma 6.4 since there are small graphs G of even size with $s(G) = 1$. One such graph is the tree T depicted in Figure 4(a), for which $s(T) = 1$. Indeed, an optimal strategy for Alice is to first colour the edge whose ends have degrees 1 and 3, resp., and then, she applies the pairing strategy with the other two edges with ends of degree 1 paired together, and the other two edges incident to the vertex of degree 3 paired together. Similarly, Bob can prevent Alice from winning by 2 or more by employing the same pairing strategy, and pairing the two edges not in the pairing for Alice's strategy, and thus, $s(T) = 1$. By starting from this tree, and repeatedly picking a vertex v and attaching to v two pending paths P and Q with lengths verifying one of the conditions from the lemma's statement, we get, at each step, another graph (actually tree) T' of even size with $s(T') = 1$. □

Through another application of Lemma 6.4, we can also fill in one of the last cells of Table 1 that our previous results do not allow to complete. That is:

Theorem 6.6. *There exist arbitrarily large graphs G of even size with $s(G) = 2$.*

Proof. The proof is similar to that of Theorem 6.5, yet a bit more involved. Start from a graph H of odd size with $s(H) = 0$. Such a graph exists, as remarked in Table 1. Let v be any vertex of the graph, and attach to v two paths P and Q with even lengths p and q, respectively, at least 4. By Lemma 6.4, note that, for the resulting graph H' which has odd size, $s(H') = 0$. Furthermore, it can be checked, in the strategy described for Bob in that lemma's proof, that if we construct the pairing $\mathcal{P}$ so that it contains the pairs $\{e_2, e_3\}$ and $\{f_2, f_3\}$, then, because $p, q \geq 4$, Bob has a drawing strategy in H' by which the common end of e_2 and e_3 (call it a) and the common end of f_2 and f_3 (call it b), get captured by no player once the game ends.

Let G be the graph obtained from H' by adding the edge ab. Note that G has even size. To see now that $s(G) = 2$, it suffices to consider the following strategy for Alice:

- During the first turn, Alice colours ab.
- From this point on, Alice reacts to Bob's moves, following the drawing strategy above in H' by which both a and b get captured by none of the players.

As a result, the game ends in a draw in H' and the vertices a and b do not get captured by either of the players in H'. The fact that Alice coloured ab during the first round then guarantees that she captures both a and b in G, thereby making her score 2 larger than Bob's score. □

In the case of trees now, Lemma 6.4 implies that, when investigating the Edge-Balanced Index Game on a given tree T, we can actually focus on its *core* $C(T)$, being the tree obtained from T by repeatedly (for as long as possible) contracting pending paths P and Q that are attached at a same vertex and verify one of the length conditions in Lemma 6.4. Specifically, in the context of Conjecture 6.1, note that, by this transformation, any tree T and its corresponding core $C(T)$ have the same size parity. Through these observations, we can now confirm Conjecture 6.1 for subdivided stars.

Theorem 6.7. *Subdivided stars comply with Conjecture 6.1, and there is a linear-time algorithm that calculates the outcome of the game in subdivided stars.*

Proof. Let T be a subdivided star. By Lemma 6.4, the outcome of the game in T is the same as the outcome in its core $C(T)$. Abusing the notation, we refer to $C(T)$ as T.

If T is a path, then the result follows from Theorem 5.1. Thus, we can assume that T has a unique vertex r of degree at least 3, to which are attached at least three pending paths. Since the converse of Lemma 6.4 cannot be applied more onto T, we deduce that there are precisely three pending paths attached to r. More precisely, r is incident to a pending path P of even length $p \geq 2$ and to a pending path Q of odd length $q \geq 3$, and is adjacent to a leaf r' (rr' being the third pending path, thus of length 1).

Note that T has even size. We claim that, in this particular setting, $s(T) = 1$. First, we prove that $s(T) < 2$. Consider the strategy for Bob where he colours an edge of P or Q incident to a leaf during the first round, and then colours edges arbitrarily afterwards. There are two cases depending on whether Alice coloured rr' at some point or not.

- If Alice coloured rr', then Bob actually coloured one edge more than Alice in the path R induced by the edges of P and Q (note that R has odd size $p+q$). By similar arguments to those used in the proof of Lemma 6.4, this implies that Bob's score is at least 1 larger than Alice's score when restricted to R. In T, the fact that Alice coloured rr' implies that she captures r'. Regarding r, the worst situation is when, in R, the vertex r is captured by none of the players, in which case r actually gets captured by Alice in T. In this case, the actual difference between the scores of Alice and Bob increases, in total, by exactly 2. Thus, Alice achieves a score at most 1 larger than Bob's score.

- If Bob coloured rr', then Alice actually coloured one edge more than Bob in the path R induced by the edges of P and Q (note that R has odd size $p + q$). By similar arguments to those used in the proof of Lemma 6.4, this implies that Alice's score is at most 1 larger than Bob's score when restricted to R (since Bob coloured an edge of P or Q incident to a leaf during the first round). In T, the fact that Bob coloured rr' implies that he captures r'. Regarding r, the worst situation is when, in R, the vertex r is captured by Alice, in which case r actually gets captured by Alice in T. In this case, Bob achieves at least the same score as Alice.

Hence, $s(T) < 2$. To see that $s(T) > 0$, and thus, $s(T) = 1$, consider the following strategy for Alice. Alice colours rr' in the first round. Now, whenever Bob colours an edge incident to r, Alice colours the other edge incident to r, and whenever Bob colours an edge in R that is incident to a leaf, Alice colours the other edge in R that is incident to a leaf. Otherwise, Alice colours any arbitrary edge. By similar arguments to those used in the proof of Lemma 6.4 and just above for Bob's strategy, we get that, in R, Bob's score is exactly 1 larger than Alice's score and neither player captures r, but, in T, Alice captures r' and r, and thus, Alice's score is 1 larger than Bob's score in T.

The linear-time algorithm follows by repeatedly applying Lemma 6.4 to obtain the core $C(T)$, and then the result follows by Theorem 5.1 if $C(T)$ is a path, and otherwise, $s(T) = 1$ as was shown above. □

This idea of reducing a tree to studying its core can be of further use in understanding other tree classes. For instance, through this approach, studying Conjecture 6.1 in the context of caterpillars can be reduced down to studying the problem for caterpillars of maximum degree 3 only. Understanding this narrowed class of trees, however, remains an interesting challenge at this point. Through computer experimentations, we were notably able to observe interesting phenomena. For instance, for the caterpillar T' displayed in Figure 4(b), $s(T') = 2$, and T' has the intriguing property that, to win a game with a score 2 larger than Bob's score, Alice must colour the bolded dashed edge on her first turn. In other words, if Alice colours any other edge of that tree on her first turn, then Bob has a strategy to guarantee his score is at most 1 smaller than Alice's score. Several examples, including this particular one, show that, contrarily to what one could think, Alice colouring any edge incident to a leaf during her first turn does not always guarantee the best score possible for Alice.

7. Outcome of the game in complete graphs

In this section, we study the game in complete graphs. A surprising fact is that, despite complete graphs being the graphs with the most symmetrical structure, our tools from Section 3 do not apply to them. Due to the inherent difficulty of the game in complete graphs, we only exhibit partial results which, in combination with Lemma 6.4, help fill Table 1, and we hope that these results can serve as a stepping stone towards resolving the game in complete graphs in the future. We begin with the following corollary which is a direct consequence of results from previous sections.

Corollary 7.1. *Let $n \geq 2$. Then,*

$$s(K_n) \in \begin{cases} \{0,2\} & \text{if } n \text{ is even,} \\ \{0,1\} & \text{otherwise.} \end{cases}$$

Proof. This follows from Corollary 4.1 and Observation 4.2 since every vertex of K_n has degree $n-1$. □

Corollary 7.1, the results to follow, and numerous attempts to resolve the game for complete graphs lead us to believe the following conjecture is true.

Conjecture 7.2. *Let $n \geq 2$. Then,*

$$s(K_n) = \begin{cases} 2 & \text{if } n = 2, \\ 1 & \text{if } n \equiv 3 \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

We begin by proving that $s(K_4) = 0$. We then use this result to prove that $s(K_5) = 0$, in order to illustrate the technique we were attempting to use to prove that $s(K_{n+4}) = 0$ if $s(K_n) = 0$. Finally, we prove that $s(K_6) = 0$, which we use in combination with Lemma 6.4 to finish filling Table 1.

Observation 7.3. $s(K_4) = 0$.

Proof. We give a drawing strategy for Bob in K_4. Let $V(K_4) = \{v_1, v_2, v_3, v_4\}$. Bob's strategy is as follows. W.l.o.g., let v_1v_2 be the first edge Alice colours. Then, Bob colours v_3v_4. If Alice colours the edge v_1v_3 (v_2v_3, resp.), then Bob colours the edge v_2v_3 (v_1v_3, resp.). If Alice colours the edge v_1v_4 (v_2v_4, resp.), then Bob colours the edge v_2v_4 (v_1v_4, resp.). By this strategy, Bob captures at least v_3 and v_4, and hence, guarantees at least a draw. Thus, $s(K_4) = 0$. □

Observation 7.4. $s(K_5) = 0$.

Proof. We give a drawing strategy for Bob in K_5. Let $V(K_5) = \{v_1, \ldots, v_5\}$. Bob's strategy is as follows. W.l.o.g., let v_1v_2 be the first edge Alice colours. Then, Bob colours v_3v_4. The general idea is that Bob will play his drawing strategy from Observation 7.3 (or a winning strategy if Alice does not play optimally) in the K_4 induced by the vertices v_1, v_2, v_3, and v_4. This ensures that v_3 and v_4 have two incident blue edges and one incident red edge in the K_4. Thus, whenever Alice plays in the K_4 induced by the vertices v_1, v_2, v_3, and v_4, then Bob responds with his drawing strategy in the K_4. Whenever Alice colours the edge v_1v_5 (v_3v_5, resp.), Bob colours the edge v_3v_5 (v_1v_5, resp.). Also, whenever Alice colours the edge v_2v_5 (v_4v_5, resp.), Bob colours the edge v_4v_5 (v_2v_5, resp.). The vertices that are captured by the players now depends on how Alice played in the K_4. There are two cases:

1. v_1 and v_2 have two incident red edges and one incident blue edge in the K_4.
2. one of v_1 and v_2, say v_1, has three incident red edges and no incident blue edges, while v_2 has two incident blue edges and one incident red edge in the K_4.

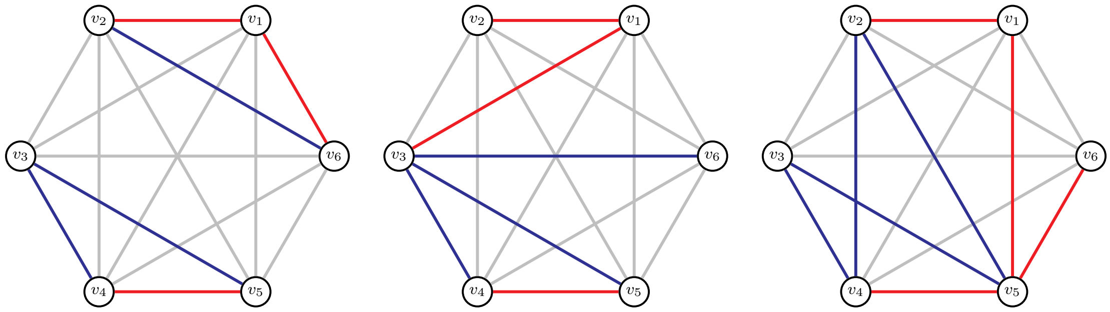
Let $j \in \{1, 2, 3, 4\}$ and let $\overline{j} = j - 2$ if $j \in \{3, 4\}$, and $\overline{j} = j + 2$ otherwise. Since Bob plays optimally in the K_4, and, for every red edge of the form v_jv_5, there is a blue edge of the form $v_{\overline{j}}v_5$, in the first case above, either v_3 is captured by Bob or both v_1 and v_3 are captured by neither player, and either v_4 is captured by Bob or both v_2 and v_4 are captured by neither player. In the second case above, one of v_2 and v_4 is captured by Bob, but the other is captured by neither player, and at most one of v_1 and v_3 is captured by Alice. Since, in both cases, for every red edge of the form v_jv_5, there is a blue edge of the form $v_{\overline{j}}v_5$, the vertex v_5 is captured by neither player. Hence, Bob captures at least the same number of vertices as Alice in K_5, and so, $s(K_5) = 0$. ☐

In general, the technique we tried to employ to prove that $s(K_{n+4}) = 0$ if $s(K_n) = 0$ was the same as the one in the proof of Observation 7.4. The idea was to extract a K_4 from K_{n+4}, and then have Bob play optimally in the K_4 (K_n, resp.) when Alice played in the K_4 (K_n, resp.), and have Bob pair the edges going from the K_4 to the K_n as in the proof of Observation 7.4, *i.e.*, to the same vertex in the K_n, so that the edges from the K_4 to the K_n do not affect which player (if any) captures the vertices in the K_n. In particular, if Alice is forced to play optimally in the K_4, *i.e.*, she must capture exactly 2 of the vertices of the K_4 when only counting the edges of the K_4 itself, then the result holds. However, the technique fails when Alice does not play optimally in the K_4, since she can ensure there is a vertex in the K_4 with a red degree of 3 and a blue degree of 0 when only counting the edges of the K_4. Then, she can exploit the pairing that Bob chose for the edges going from the K_4 to the K_n, in order to capture 3 of the 4 vertices in the K_4. We now move on to proving that $s(K_6) = 0$, with the next observation being useful in doing so.

Observation 7.5. *For any graph G and any integer $\ell > 0$, if*

1. *it is Alice's turn,*
2. *every edge in the subgraph induced by the vertices $v_1, \ldots, v_\ell \in V(G)$ is coloured, and*
3. *each of $v_1, \ldots, v_\ell$ is of odd degree and has a larger blue degree than red degree,*

then Bob has a strategy to ensure that he captures $v_1, \ldots, v_\ell$.



(a) Case 1 (b) Case 2.1 (c) Case 3.1

Figure 5: Some configurations reached in the proof of Proposition 7.6.

Proof. Bob's strategy is as follows. Whenever Alice colours an edge v_iv_j for integers $\ell+1 \leq i \leq |V(G)|$ and $1 \leq j \leq \ell$, Bob colours an edge v_xv_j for an integer $\ell+1 \leq x \leq |V(G)|$ such that v_xv_j is uncoloured. Whenever Alice colours any other edge, that edge is not incident to any vertex v_j (by 2.), and Bob colours any arbitrary uncoloured edge. Since $v_1, \dots, v_\ell$ all have odd degree, either Bob can follow his strategy (in which case, after Bob plays, the conditions 1., 2., and 3. all hold again) or Alice coloured the last edge incident to a vertex v_j, but, in the latter case, it must be that the vertex v_j has a larger blue degree than red degree since v_j is of odd degree and, prior to Alice's move, v_j has a larger blue degree than red degree (by 3.). Hence, Bob always captures $v_1, \dots, v_\ell$ with this strategy. □

Proposition 7.6. $s(K_6) = 0$.

Proof. We give a drawing strategy for Bob in K_6. Let $V(K_6) = \{v_1, \dots, v_6\}$. Bob's strategy is as follows. W.l.o.g., let v_1v_2 be the first edge Alice colours. Then, Bob colours v_3v_4. By symmetry, Alice only has four possible options for her next turn. Thus, w.l.o.g., either she colours v_1v_6 or v_1v_3 or v_5v_6 or v_3v_5. In each of the first three cases, Bob then colours v_3v_5. In the last case, Bob colours v_4v_6, and thus, the last case is symmetric to the third case where Alice coloured v_5v_6 (after Bob's move in each case). Hence, we may assume that there are only the first three cases by symmetry, and that Bob coloured v_3v_5 in each of these cases. We now distinguish the three cases:

Case 1: Alice coloured v_1v_6 on her second turn. Alice now colours v_4v_5, since, otherwise, Bob then colours v_4v_5, and by Observation 7.5, Bob has a strategy to capture v_3, v_4, and v_5, and thus, ensure at least a draw. Then, Bob colours v_2v_6. Now, v_2, v_4, v_5, and v_6 all have the same blue and red degrees (see Figure 5(a)). Whenever Alice colours the edge

- v_2v_4 (v_2v_5, resp.), Bob colours the edge v_2v_5 (v_2v_4, resp.).
- v_6v_4 (v_6v_5, resp.), Bob colours the edge v_6v_5 (v_6v_4, resp.).
- v_1v_4 (v_1v_5, resp.), Bob colours the edge v_1v_5 (v_1v_4, resp.).
- v_3v_2 (v_3v_6, resp.), Bob colours the edge v_3v_6 (v_3v_2, resp.).
- v_1v_3, Bob colours any arbitrary uncoloured edge.

If, at any point, the edge Bob wishes to colour is already coloured (and so, must be blue), then he colours any arbitrary uncoloured edge. After all the edges of K_6 are coloured, if we disregard the edges v_1v_4, v_1v_5, v_3v_2, and v_3v_6, then the respective blue degrees of both v_2 and v_6 are at least as large as their respective red degrees, and either the respective blue degrees of v_4 and v_5 are at least as large as their respective red degrees or one of v_4 and v_5 has a blue degree of at least three, and so, is captured by Bob. Hence, by Bob's strategy, he captures at least one of v_2 and v_6, at least one of v_4 and v_5, and he captures v_3, and so, he captures at least 3 vertices, ensuring at least a draw.

Case 2: Alice coloured v_1v_3 on her second turn. Alice must either colour v_4v_5 or an edge incident to v_3, as, otherwise, Bob can ensure at least a draw by Observation 7.5 by colouring v_4v_5 on his next turn. There are two subcases to be considered:

Case 2.1: Alice coloured v_4v_5 on her third turn. Bob then colours v_3v_6, thereby capturing v_3 (see Figure 5(b)). If Alice colours an edge incident to v_5 next, then Bob colours the edge v_6v_4. Otherwise, for any other edge Alice colours next, Bob colours the edge v_6v_5. Regardless of what Alice did on her fourth turn, one of v_6v_4 and v_6v_5 is blue after Bob's fourth turn, say, w.l.o.g., v_6v_5. Then, by Observation 7.5, Bob has a strategy to ensure capturing v_6 and v_5, and since he has already captured v_3, he ensures at least a draw.

Case 2.2: Alice coloured an edge incident to v_3 on her third turn. Hence, she either coloured the edge v_2v_3 or v_6v_3. In either case, Bob colours the remaining uncoloured edge incident to v_3, thereby capturing v_3. If Alice coloured v_2v_3, then, regardless of the edge Alice colours next, at least one of v_4v_5 and v_4v_6 is uncoloured, and Bob colours one of them that is uncoloured on his next turn, say, w.l.o.g., v_4v_5. Then, by Observation 7.5, Bob can ensure capturing v_4 and v_5 (v_4 and v_6 if Bob coloured v_4v_6 on his previous turn), and since he has already captured v_3, he ensures at least a draw.

Hence, Alice must have coloured v_6v_3 on her third turn. Alice then colours v_4v_5 on her fourth turn, since, otherwise, Bob colours v_4v_5, and can ensure capturing both v_4 and v_5 by Observation 7.5, and thus, ensure at least a draw. Bob then colours v_2v_4, and can ensure capturing both v_2 and v_4 by Observation 7.5, and thus, ensure at least a draw.

Case 3: Alice coloured v_5v_6 on her second turn. Alice must colour an edge incident to v_5, as, otherwise, Bob can ensure at least a draw by Observation 7.5 by colouring v_4v_5 on his next turn. There are two subcases to be considered:

Case 3.1: Since Alice colouring v_1v_5 or v_2v_5 is symmetric in this case, we can assume that Alice coloured v_1v_5 on her third turn. Bob then colours v_2v_5. Alice then colours v_4v_5 on her fourth turn, since, otherwise, Bob colours v_4v_5, thereby capturing v_5, and then Bob can ensure capturing v_3 and v_4 by Observation 7.5, and thus, ensure at least a draw. Bob then colours v_2v_4 (see Figure 5(c)). Now, whenever Alice colours an edge incident to v_4, Bob colours another uncoloured edge incident to v_4. If Alice colours an edge incident to v_2 and/or v_3, then Bob colours v_2v_3 if possible, or else an edge incident to v_2 (the case where Alice coloured v_2v_3). Bob colours v_2v_3 in any other case. Thus, by Observation 7.5, Bob can ensure capturing at least v_2, v_3, and v_4, and thus, ensure at least a draw.

Case 3.2: Alice coloured v_4v_5 on her third turn. Bob then colours v_2v_4. If Alice now colours v_1v_3, v_1v_5, v_1v_6, or v_3v_6, then Bob colours v_2v_3, and can ensure capturing v_2, v_3, v_4 by Observation 7.5, and hence, ensure at least a draw. There are three subcases to be considered:

Case 3.2.1: Alice coloured v_1v_4 or v_6v_4 on her fourth turn. Bob then colours v_2v_5. Now, whenever Alice colours the edge v_2v_3 or an edge incident to v_2, then Bob colours v_2v_3 if possible, or else an edge incident to v_2. Whenever Alice colours an edge incident to v_3 that is not v_2v_3, then Bob colours another uncoloured edge incident to v_3. If Alice colours the last edge incident to v_4 (v_5, resp.), then Bob colours the last edge incident to v_5 (v_4, resp.). Bob colours any arbitrary uncoloured edge in any other case. By this strategy, Bob ensures capturing v_2, v_3, and at least one of v_4 and v_5, and hence, ensures at least a draw.

Case 3.2.2: Alice coloured v_6v_2 or v_2v_3 on her fourth turn. Bob then colours v_2v_5. Now, if Alice colours the last edge incident to v_2 (v_5, resp.), then Bob colours the last edge incident to v_5 (v_2, resp.). Whenever Alice colours an edge incident to v_4, Bob colours an uncoloured edge incident to v_4. If Alice colours v_1v_3 (v_6v_3, resp.), then Bob colours v_6v_3 (v_1v_3, resp.). Bob colours any arbitrary uncoloured edge in any other case. By this strategy, Bob ensures capturing v_3, v_4, and at least one of v_2 and v_5, and hence, ensures at least a draw.

Case 3.2.3: Alice coloured v_2v_5 on her fourth turn. Bob then colours v_1v_6. Now, whenever Alice colours v_1v_4 (v_6v_4, resp.), Bob colours v_6v_4 (v_1v_4, resp.). Whenever Alice colours v_1v_3 (v_6v_3, resp.), Bob colours v_6v_3 (v_1v_3, resp.). Whenever Alice colours v_2v_6 (v_1v_5, resp.), Bob colours v_1v_5 (v_2v_6, resp.). Bob colours any arbitrary uncoloured edge in any other case. By this strategy, Bob ensures capturing v_3, v_4, and at least one of v_1 and v_6, and hence, ensures at least a draw. □

We have proven that there exists a small graph G of odd size with $s(G) = 0$, *i.e.*, K_6. Combining this result with Lemma 6.4 gives us the following:

Theorem 7.7. *There exist arbitrarily large graphs G of odd size such that $s(G) = 0$.*

Proof. This follows from Lemma 6.4 since $s(K_6) = 0$ by Proposition 7.6. By starting from K_6, and repeatedly picking a vertex v and attaching to v two pending paths P and Q with lengths verifying one of the conditions from the statement of Lemma 6.4, we get, at each step, another graph G of odd size such that $s(G) = 0$. □

8. Conclusion

In this paper, we have studied several aspects of the Edge-Balanced Index Game, which, although it was mentioned in [5], had received no dedicated attention prior to the current work. We gave several results related both to the general behaviour and main mechanisms behind the game, and to understanding the game in common graph classes. While some of our results are rather expected for such an impartial scoring game, like the outcome results from Section 2, some others establish its very own peculiarities and subtleties, such as the behavioural results from Section 4.

From a global look, the game seems rather hard to comprehend. This is attested, notably, by its general instability. For instance, understanding the game on a given graph does not guarantee anything regarding its supergraphs, even for those that are very close. Recall, indeed, that there are graphs G with $s(G) = 0$, for which the graph G' obtained by adding an edge to G is such that $s(G') = 2$. Another illustration of the hardness of the game is the different types of dedicated arguments we had to develop to understand it in graph classes that are, sometimes, very simple. The case of trees seems particularly intricate, and an even more intriguing case is that of complete graphs, for which one could think the tools developed in Section 3 should have been a perfect fit.

Three directions for further research on the topic seem particularly interesting to us. The first one is Conjecture 6.1, which, despite the several tools and partial results we came up with, we have not been able to prove. More generally speaking, it would be interesting to investigate whether Corollary 4.7 extends to trees or not. The second direction concerns Conjecture 7.2. This direction is particularly intriguing since the tools we have developed seem insufficient to deal with complete graphs, and so, a new technique should be necessary to resolve the game in complete graphs, which may be of interest on its own. The last direction we have in mind is the general complexity of the game. In particular, we wonder whether the outcome of the game on a given graph can be decided in polynomial time or if it is hard for some complexity class. One difficulty we experienced while trying to design hardness reductions is that the game progresses at a very slow pace, which makes it hard to force a game to follow an anticipated scenario. For such reasons, and the fact that games played on the edges of graphs, such as the Shannon switching game [3], sometimes show up to be polynomial-time solvable, we would not be too surprised if the Edge-Balanced Index Game turned out to be polynomial-time solvable as well.

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